UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

LIBRARY ROUTINE FA 1 - 122

TITLE — Second Order Linear Differential Equation with Two Point Boundary Conditions (DOI Only)

NUMBER OF WORDS — 101

TYPE — Closed auxiliary for code Al; Code XAl, the floating point constant listing auxiliary, must also be used. The routine is entered as follows:

p	50 gF
	50 pF
p + 1	26 xF

where g is the location of the auxiliary routine and x is the location of Code FA 1.

TEMPORARY STORAGE — See parameters.

DURATION — $T = (n + 1)(t + 400) + 400$ approximately where T is the total time in milliseconds, t is the time in milliseconds of the auxiliary routine and n is the number of sub-intervals into which the range of x is divided.

ACCURACY — 9 decimal places or less, depending on the condition of the given equation.

PARAMETERS — Preset parameters are S3, S4, S5, S6, S7.

LOCATION	CONTENT	USE
3	00F 00 mF	m is the location of the floating point accumulator.
4	00F 00 ℓF	ℓ is the location of code Al
5	00F 00 kF	Locations k + j (j = 0, ..., 2n +3) are used for temporary storage. The solutions y_i (i = 0, ... , n) will appear in the n + 1 successive locations k + n + 3 to k + 2n + 3.

6	00F 00 qF	Locations q + j (j = 0, 1,...,10) contain the difference equation coefficients a_i, b_i, c_i, d_i, the differential equation coefficients, A_i, B_i, C_i, D_i, (i = 0, 1, ... , n), the increment Δ (x) (8S6), and a temporary storage (9S6). The differential equation coefficients, A_i, B_i, C_i, D_i, calculated by the auxiliary
	ro	routine, must be stored in locations 4S6, 5S6, 6S6, and 7S6 respectively.
7	00F 00 rF	Locations r+j (j = 0, 1, ..., 7) must contain the length of the range of x, L, and the boundary equation coefficients E, F, G, H, K, M expressed as floating point numbers, at the start of code FA 1. The quantities L, E, F, G, H, K and M are to be stored in locations S7, 1S7, 2S7, 3S7, 4S7, 5S7, and 6S7 respectively.

PROGRAM PARAMETER

8	00F 00 nF	n is the number of sub-intervals into which the range of x is divided.

DESCRIPTION Over an interval of length L at equally spaced points, x_i (i = 0, 1, ... ,n), this routine will give the solution of the differential equation,

$$Ay'' + By' + Cy = D,$$

with the given boundary conditions,

$$Ey' + Fy = G]_{x=x_0}$$

and

$$Hy' + Ky = M]_{x=x_m = n\Delta x + x_0}$$

where A, B, C, D, E, F, G, H, K and M are functions of x, the independent variable.

An open subroutine, written using interpretive orders, must be provided starting at location g. It is to evaluate the coefficients A, B, C and D for a given x, $x_0 \leq x \leq n\,\Delta x + x_0$, of the differential equation. These calculated coefficients A, B, C and D must be stored successively starting at Location 4S6. This auxiliary subroutine must be left with the word

8K 1F

83 (address of the 8th word of Code Fal) F.

NOTE: Δx is available for use in the auxiliary subroutine. Δx is stored in 8S6.

METHOD: This routine first calculates the coefficients of the difference equations,

$$a_i y_{i+1} + b_i y_i + c_i y_{i-1} = d_i \qquad i \neq 0,n$$

$$a_0 y_1 + b_0 y_0 = d_0$$

$$b_n y_n + c_n y_{n-1} = d_n$$

from the coefficients of the differential equations. The following equations are used to calculate a_i, b_i, c_i and d_i $(0 \leq i \leq n)$:

$$a_i = A_i + B_i\,\Delta x/2$$

$$b_i = C_i \cdot \Delta x^2 - 2\,A_i \qquad \text{for } i \neq 0,\ n$$

$$c_i = A_i - B_i\,\Delta x/2 \qquad \Delta x = L/n$$

$$d_i = D_i\,\Delta x^2$$

$$a_0 = A_0 E_0$$

$$b_0 = (E_0 C_0 - F_0 B_0) \cdot \Delta x^2/2 + A_0 F_0 \cdot \Delta x - A_0 E_0$$

$$c_0 = 0$$

$$d_0 = A_0 G_0 \Delta x + (E_0 D_0 - B_0 G_0) \Delta x^2/2$$

$$a_n = 0$$

$$b_n = (K_n B_n - C_n H_n) \Delta x^2/2 + A_n H_n + A_n K_n \Delta x$$

$$c_n = -A_n H_n$$

$$d_n = A_n M_n \cdot \Delta x + (B_n M_n - H_n D_n) \Delta x^2/2$$

At each step, these coefficients are stored in the four locations starting with location q. Then this routine evaluates the following formulae:

$$\beta_i = b_i - c_i \cdot \nu_{i-1}$$

$$\tau_i = d_i - c_i \quad \tau_{i-1} / \beta_i$$

$$\nu_i = a_i / \beta_i$$

when i = 0, ... ,n and $\nu_{-1} = 0 = \tau_{-1} \cdot \beta_0$ is calculated first and stored in location k + 1. Then τ_0 is calculated and stored in location k + n + 3. Finally ν_0 is calculated and stored over β_0 in location k + 1. During the next time thru the loop β_1, τ_1 and ν_1 are calculated and stored in locations k + 2, k + n + 4 and k + 2 resprectively. This process is continued until β_n, τ_n and ν_n have been calculated and stored. At this stage, the τ_i are stored in locations k + 1 thru k + n + 1 and the τ_i, are stored in locations k + n + 3 thru k + 2n + 3. When this has been done, the routine calculates the solution using the formula

$$y_i = \tau_i - \nu_i \tau_{i+1} \qquad i = n, \ldots, 0.$$

where $y_{n+1} = 0$. These solutions, y_i are stored over the τ_i in locations $k + n + 3$ thru $k + 2n + 3$.

NOTE: Incorporated within the routine is a test to determine if $\beta_i = 0$. When $\beta_i = 0$, $1/4 \times 10^{-36}$ is stored in place of this $\beta_i = 0$.

The method was adapted by S. Gill from a process used by L. Fox (L. Fox and H. H. Robertson; Automatic Digital Computation Symposium, National Physical Laboratory, England, March 1953. Paper 19, pp. 5, 6).

RT: 9/27/55
DATE February 16, 1954
PROGRAMMED BY R. Polivka
APPROVED BY J. Nash

RP: mge
9/27/55

LOCATION	ORDER		NOTES PAGE 1 FA 1
0	K5 29F		
	42 97L		Set up link and address of
1	10 20F		auxiliary subroutine
	42 7L		
2	36 83L		
	26 83L		
3	41 S5		
	50 3L		
4	26 S4		
	8K S8		
5	8S 8S6		
	85 S7		Calculate Δx
6	86 8S6		
	8S 8S6		
7	8K 1F	From 33, 34	to auxiliary subroutine
	83 (g)F	By 1	
8	(85 4S6	By 74, 78	replaced by 8K 1F
	87 1S7)		83 (54)L
9	8S S6		Store a_0
	81 5S6		
10	87 2S7		
	8S 9S6		
11	85 6S6		
	87 1S7		
12	84 9S6		
	87 8S6		
13	87 8S6		Calculate the initial difference
	87 n+5+00		
14	80 S6		equation coefficients, a_0, b_0, c_0
	8S 9S6		and d_0
15	85 4S6		
	87 8S6		
16	87 2S7		
	84 9S6		

LOCATION	ORDER		NOTES PAGE 2
17	8S 1S6		store b_0
	8K 0F		
18	8S 2S6		store c_0
	81 5S6		
19	87 3S7		
	8S 9S6		
20	85 1S7		
	87 7S6		
21	84 9S6		
	87 8S6		
22	87 8S6		
	87 N+5+00		
23	8S 9S6		
	85 4S6		
24	87 8S6		
	87 3S7		
25	84 9S6		
	8S 3S6		store d_0
26	8J 74L		
	0K S8		
27	81 2S6	From 64,53	
	07 S5		
28	84 1S6		
	8J 69L		
29	0S 1S5		store β_i
	81 2S6		Calculate β_i, τ_i, and ν_i.
30	07 (k+n+2)F		
	84 3S6		
31	06 1S5		
	0S (k+n+3)F		store τ_i
32	85 S6		
	06 1S5		

LOCATION	ORDER		NOTES PAGE 3	FA 1
33	0S 1S5		store v_i	
	03 7L			
34	8J 77L			
	83 (7)L	by 82		
35	(8K 0F		replaced by 8K 1F	
	8S S6)		82 64L	
36	81 4S6			
	87 4S7			
37	8S 2S6			
	81 6S6			
38	87 4S7			
	8S 9S6			
39	85 5S6			
	87 5S7			
40	84 9S6			
	87 8S6			
41	87 8S6			
	87 N+5+00			
42	80 2S6			
	8S 9S6			
43	85 4S6			
	87 8S6			
44	87 5S7			
	84 9S6			
45	8S 1S6		Calculate to the difference	
	81 4S7		equation coefficients a_n, b_n	
46	87 7S6		c_n and d_n.	
	8S 9S6			
47	85 5S6			
	87 6S7			
48	84 9S6			
	87 8S6			

LOCATION	ORDER		NOTES PAGE 4
49	87 8S6		
	87 N+5+00		
50	8S 9S6		
	85 4S6		
51	87 8S6		
	87 6S7		
52	84 9S6		
	8S 3S6		
53	8K 1F		
	83 27L		
54	85 5S6	from 8	
	87 8S6		
55	87 N+5+00		
	8S 9S6		
56	84 4S6		
	8S S6		store a_i
57	81 9S6		
	84 4S6		
58	8S 2S6		store
	85 6S6		calculate the difference
59	87 8S6		equation coefficients a_i, b_i,
	87 8S6		c_i and d_i $i \neq 0,n$.
60	80 4S6		
	80 4S6		
61	8S 1S6		store b_i
	85 7S6		
62	87 8S6		
	87 8S6		
63	8S 3S6		store d_i
	8K 1F		
64	83 27L		
	1K S8	From 35	
65	11 (k+2n+3)F		
	17 (k+n)F		

LOCATION	ORDER		NOTES PAGE 5 FA 1
66	14 (k+2n+2)F		Calculate the solutions, y_i
	1S (k+2n+2)F		
67	1L 1022F		
	13 65L		
68	8J 94L		
	32 94L		
69	L3 S3	from 28	
	32 70L		
70	26 29S4		
	19 1F		
71	40 S3		Test for $\beta_i = 0$
	L5 73L		
72	40 1S3		
	26 29S4		
73	00 F		
	00 28F		
74	L5 76L	From 26	
	40 8L		Replace word 8
75	26 29S4		by 8K 1F 83 54L
	00 7L		
76	8K 1F		
	83 54L		
77	49 S3	From 34	
	L5 93L		
78	42 8L		Switch, 1st time go to part of
	L1 98L		code to calculate a_n, b_n, c_n, d_n.
79	40 98L		2nd time,
	36 29S4		
80	L5 100L		Go to part of code to calculate
	40 35L		the y_i
81	L5 93L		
	42 34L		

LOCATION	ORDER		NOTES
82	26 29S4		
	00 F		waste
83	L5 5F	From 2	
	L4 8F		
84	42 65L		
	F5 65L		
85	L4 70L		
	42 92L		
86	00 20F		
	46 30L		Set addresses in routine dependent
87	F5 65L		
	F4 70L		on the program parameters.
88	42 31L		
	L5 92L		
89	L4 8F		
	42 66L		
90	00 20F		
	46 66L		
91	F5 66L		
	00 20F		
92	46 65L		
	41 (k+n+2)F	By 85	
93	26 3L		
	00 35L		
94	L5 98L		
	40 8L		
95	L5 99L		Reset words
	40 35L		
96	L5 75L		8, 34, and 35
	42 34L		
97	32 97L		
	22 ( )F	By 0	link

LOCATION	ORDER	NOTES	PAGE 7	FA 1
98	85 4S6			
	87 1S7			
99	8K 0F			
	8S S6			
100	8K 1F			
	82 64L			

RT:
SEPT. 28, 1955
MGE